JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 113, 36–58 (1986)

On Discrete Stochastic Bilinear Systems Stability

C. S. KUBRUSLY*

Department of Research and Development, National Laboratory for Scientific Computation-LNCC/CNPq, Rua Lauro Müller 455, 22290-Rio de Janeiro-RJ, Brazil
and
Department of Electrical Engineering, Catholic University-PUC/RJ, Rua Marques de São Vicente 209, 22453-Rio de Janeiro-RJ, Brazil

Submitted by Harold Kushner

Mean square stability conditions for discrete-time bilinear systems operating in a stochastic environment are given in this paper. Only independence and wide sense stationarity are required for the second order disturbance sequences involved, thus dismissing ergodicity and zero-mean assumptions. Stochastic stability conditions are derived by using a deterministic stability result for a class of separable nonlinear dynamical systems evolving in a Banach space. © 1986 Academic Press, Inc.

1. INTRODUCTION

Bilinear systems comprise an important subclass of nonlinear dynamical systems, which lately experienced a remarkable research effort. The available results have reached a certain level of maturity which has already justified some systematic presentation in book form [1, 2]. More recent achievements in both theory and applications of bilinear systems can presently be found in some general surveys [3–5] and special issues [6–9].

Stability for continuous-time bilinear systems operating in a stochastic environment has been investigated by several authors. Both the stochastic Liapunov function approach, based on a general (either linear or non-linear) stochastic stability theory [10, 11], or the Lie-algebraic approach for analysing moments stability [12], have been extensively used in the current literature (e.g., see the survey in [5] and the special issue [13]). For infinite dimensional systems the problem was approached in a Hilbert space setting in [14, 15].

On the other hand, little has been written on stability for discrete-time

* This research was partially supported by CNPq (Brazilian Research Council) under Grant 404693/83–MA.

0022-247X/86 $3.00

bilinear systems operating in a stochastic environment, compared with what has been done for the continuous-time case. Some frequency domain results were considered in [16, 17] for particular classes of single-input models, and almost sure and mean square stability conditions were presented in [18]. Conditions for mean square stability were also obtained in [19] towards the bilinear system identification problem.

The purpose of the present paper is to investigate mean square stability conditions for discrete dynamical systems of the form

$$x(i+1)=\left[A_0+\sum_{k=1}^{\mathrm{p}} A_k\omega_k(i)\right]x(i)+Bu(i),$$

where $\{A_k;\ k=0, 1,..., \mathrm{p}\}$ and B are linear transformations, and $\{w(i)=(\omega_1(i),..., \omega_{\mathrm{p}}(i));\ i\geqslant 0\}$ and $\{u(i);\ i\geqslant 0\}$ are second order random sequences. Such a model is properly described and discussed in section 3, where the evolution of the state moments $q(i)=E\{x(i)\}$ and $Q_v(i)=E\{x(i+v)\,x^*(i)\}$ is also analysed. Conditions on the above stochastic model under which the sequences $\{q(i);\ i\geqslant 0\}$ and $\{Q_v(i);\ i\geqslant 0\}$ converge are established in Section 4, by using a preliminary deterministic stability result developed in Section 2.

2. Preliminaries

In this section we pose the notation and auxiliary results which will be needed in the sequel.

Notation

If X is a normed linear space (an inner product space), then the symbol $\|\ \|$ $(\langle\ ;\ \rangle)$ will stand for norm (inner product) in X. If X and Y are normed linear spaces, then $\mathscr{B}[X, Y]$ will denote the normed linear space of all bounded linear transformations of X into Y. For simplicity we set $\mathscr{B}[X]=\mathscr{B}[X, X]$. Let $\mathbb{F}$ denote either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$, and $\mathbb{F}^n$ the n-dimensional (either real or complex) Euclidean space. A transformation in $\mathscr{B}[\mathbb{F}^{\mathrm{p}}, \mathbb{F}^n]$ will be identified with its n by p matrix representation relative to the standard orthonormal bases for $\mathbb{F}^n$ and $\mathbb{F}^{\mathrm{p}}$. For arbitrary $x=(\xi_1,..., \xi_n)\in\mathbb{F}^n$ and $y=(\upsilon_1,..., \upsilon_{\mathrm{p}})\in\mathbb{F}^{\mathrm{p}}$ the transformation $(xy^*)\in\mathscr{B}[\mathbb{F}^{\mathrm{p}}, \mathbb{F}^n]$, such that

$$(xy^*)\,z=x(y^*z)=x\langle z;y\rangle=x\sum_{k=1}^{\mathrm{p}}\zeta_k\bar{\upsilon}_k$$

for all $z=(\zeta_1,..., \zeta_{\mathrm{p}})\in\mathbb{F}^{\mathrm{p}}$, is identified with the usual outer product n by p matrix $[\xi_k\bar{\upsilon}_l]$, $k=1,..., n$, $l=1,..., \mathrm{p}$ (the overbar denoting complex conjugate, and the asterisk representing conjugate transpose). The Banach space of all $\mathbb{F}$-valued (absolutely) summable one-sided infinite sequences will be denoted by l_1 as usual. Let Z' be the complexification of a Banach space Z (e.g., $\mathbb{C}^n$ can be thought of as the complexification of $\mathbb{R}^n$), and define for each $L \in \mathscr{B}[Z]$ the following operator $L' \in \mathscr{B}[Z']$: $L' = L$ if Z is a complex space (i.e., if $Z = Z'$); and $L'(x + \sqrt{-1}\,y) = Lx + \sqrt{-1}\,Ly$ for all $x, y \in Z$, if Z is a real space. Now let $\sigma(L') \subset \mathbb{C}$ be the spectrum (i.e., the nonempty compact set of all spectral values) of $L' \in \mathscr{B}[Z']$. The (generalized) spectral radius of $L \in \mathscr{B}[Z]$ is then defined as the spectral radius of $L' \in \mathscr{B}[Z']$. That is, $r_\sigma(L) = r_\sigma(L') = \max\{|\lambda|: \lambda \in \sigma(L') \subset \mathbb{C}\}$. Roughly speaking, $r_\sigma(L)$ is the maximum of the absolute value of all spectral values of $L \in \mathscr{B}[Z]$, "including complex spectral values." We recall that

$$\lim_{i \to \infty} \|L^i\|^{1/i} = r_\sigma(L) \leqslant \|L'\| = \|L\| = \sup\{\|Lx\|: \|x\| = 1\}.$$

If U and V are Hilbert spaces, $L^* \in \mathscr{B}[U, V]$ will denote the adjoint of $L \in \mathscr{B}[V, U]$. $L \geqslant 0$ will be used if a self-adjoint (i.e., $L = L^*$) operator $L \in \mathscr{B}[V]$ is nonnegative (i.e., $\langle Lx; x\rangle \geqslant 0$; $\forall x \in V$). We set $\mathscr{B}[V]^+ = \{L \in \mathscr{B}[V]: L = L^* \geqslant 0\}$, the closed convex cone of all self-adjoint nonnegative operators in $\mathscr{B}[V]$. We finally recall that if $L \in \mathscr{B}[V]$ is normal (i.e., if $LL^* = L^*L$), then $r_\sigma(L) = \|L\| = \sup\{|\langle Lx; x\rangle|; \|x\| = 1\}$.

An evolution result

Before introducing the convergence results, which will enable us to establish stability conditions, we need to prove the following.

PROPOSITION (P-1). *Let $\{\xi_i;\ i \geqslant 0\}$, $\{v_i;\ i \geqslant 0\}$, and $\{\beta_i;\ i \geqslant 0\}$ be real sequences, and let $\{\alpha_i;\ i \geqslant 0\}$ and $\{\mu_i;\ i \geqslant 0\}$ be nonnegative real sequences. Set*

$$\phi_\delta(i, j) = \begin{cases} 1, & \text{if } i = j, \\ \prod_{\kappa=j}^{i-1} \delta_\kappa, & \text{if } i > j, \end{cases}$$

for any real sequence $\{\delta_i;\ i \geqslant 0\}$. Now consider the following inequalities:

(a) $\xi_{i+1} \leqslant (\alpha_i + \mu_i)\,\xi_i + \beta_i v_i, \qquad \forall i \geqslant 0.$

(b) $\xi_i \leqslant \phi_\alpha(i, 0)\,\xi_0 + \sum_{j=0}^{i-1} \phi_\alpha(i, j+1)[\mu_j \xi_j + \beta_j v_j], \qquad \forall i \geqslant 1.$

(c) $\xi_i \leqslant \phi_{\alpha+\mu}(i, 0)\,\xi_0 + \sum_{j=0}^{i-1} \phi_{\alpha+\mu}(i, j+1)\,\beta_j v_j, \qquad \forall i \geqslant 1.$

We claim that (a) $\Rightarrow$ (b) $\Rightarrow$ (c). *In particular, $(a') \Rightarrow (b') \Rightarrow (c')$, with (a'), (b'), and (c') standing for the following inequalities:*

(a′) $\xi_{i+1} \leqslant (\alpha + \sigma\mu)\,\xi_i + \sigma\beta v_i, \qquad \forall_i \geqslant 0,$

(b′) $\xi_i \leqslant \sigma[\alpha^i \xi_0 + \sum_{j=0}^{i-1} \alpha^{i-j-1}(\mu\xi_j + \beta v_j)], \qquad \forall i \geqslant 1,$

(c′) $\xi_i \leqslant \sigma[(\alpha+\sigma\mu)^i\, \xi_0 + \beta \sum_{j=0}^{i-1} (\alpha+\sigma\mu)^{i-j-1}\, v_j], \qquad \forall i \geqslant 1,$

where α, μ, β, *and* σ *are real constants such that* $\alpha > 0$, $\mu \geqslant 0$, $\sigma \geqslant 0$, *and* $\xi_0 \leqslant \sigma\xi_0$.

Proof. Let (a) be satisfied for all $i \geqslant 0$. It is trivially verified that (b) holds for $i = 1$. Suppose (b) is true for some $i \geqslant 1$. Then

$$\xi_{i+1} \leqslant \alpha_i \left(\phi_\alpha(i, 0)\, \xi_0 + \sum_{j=0}^{i-1} \phi_\alpha(i, j+1)[\mu_j \xi_j + \beta_j v_j] \right) + \mu_i \xi_i + \beta_i v_i$$

$$= \phi_\alpha(i+1, 0)\, \xi_0 + \sum_{j=0}^{i} \phi_\alpha(i+1, j+1)[\mu_j \xi_j + \beta_j v_j],$$

thus $(a) \Rightarrow$ (b) by induction. In particular, (a) $\Rightarrow$ (c). Now suppose $\{\xi_i;\ i \geqslant 0\}$ satisfies (b), and set $\zeta_0 = \xi_0$ and

$$\zeta_i = \phi_\alpha(i, 0)\, \xi_0 + \sum_{j=0}^{i-1} \phi_\alpha(i, j+1)[\mu_j \xi_j + \beta_j v_j], \qquad \forall i \geqslant 1,$$

such that $\xi_i \leqslant \zeta_i$ for every $i \geqslant 0$. Hence

$$\zeta_{i+1} = \alpha_i \zeta_i + \mu_i \xi_i + \beta_i v_i \leqslant (\alpha_i + \mu_i)\, \zeta_i + \beta_i v_i, \qquad \forall i \geqslant 0.$$

Therefore $\{\zeta_i;\ i \geqslant 0\}$ satisfies (a), and so it also satisfies (c). That is,

$$\xi_i \leqslant \zeta_i \leqslant \phi_{\alpha+\mu}(i, 0)\, \xi_0 + \sum_{j=0}^{i-1} \phi_{\alpha+\mu}(i, j+1)\, \beta_j v_j, \qquad \forall i \geqslant 1,$$

whenever $\{\xi_i;\ i \geqslant 0\}$ satisfies (b). Then (b) $\Rightarrow$ (c). The particular case is readily verified by setting $\alpha_0 = \sigma\alpha$, $\alpha_i = \alpha$ for every $i \geqslant 1$, $\beta_i = \sigma\beta$ and $\mu_i = \sigma\mu$ for every $i \geqslant 0$. ∎

Some Convergence Results

The purpose of the remainder of this section is to present the preliminary stability result in Lemma (L-1). This will be based on Propositions (P-1) and (P-2). We begin with two auxiliary remarks.

Remark (R-1). Let X be a Banach space and $A \in \mathscr{B}[X]$. It is well known that the following assertions are equivalent: (a) $\|A^i\| \to 0$ as $i \to \infty$ (i.e., A is *uniformly asymtotically stable*), (b) $r_\sigma(A) < 1$, and (c) there exist real constants $\sigma \geqslant 1$ and $0 < \alpha < 1$ such that $\|A^i\| \leqslant \sigma\alpha^i$ for every $i \geqslant 0$. Moreover, if $r_\sigma(A) < 1$ we can choose $\alpha = r_\sigma(A) + \varepsilon$ for any $\varepsilon \in (0, 1 - r_\sigma(A))$ (e.g. see [20]).

Remark (R-2). Let $\{x_i;\ i \geqslant 0\}$ be a sequence in a normed linear space X, and consider the following assertions.

(a) $\{x_i; i \geqslant 0\}$ is *bounded*. That is, $\sup_{i \geqslant 0} \|x_i\| < \infty$.

(b) $\{x_i; i \geqslant 0\}$ is a *null sequence*. That is, $\|x_i\| \to 0$ as $i \to \infty$.

(c) $\{x_i;\ i \geqslant 0\}$ is *absolutely summable*. That is, $\{\|x_i\|;\ i \geqslant 0\} \in l_1$.

(d) $\{x_i; i \geqslant 0\}$ has a *summable absolute envelope*. That is, $\{\sup_{\nu \geqslant 0} \|x_{i+\nu}\|;\ i \geqslant 0\} \in l_1$.

(e) $\{x_i; i \geqslant 0\}$ is a *Cauchy summable sequence*. That is, $\{\sup_{\nu \geqslant 0} \|x_{i+\nu} - x_i\|;\ i \geqslant 0\} \in l_1$.

(f) $\{x_i; i \geqslant 0\}$ is a *Cauchy sequence*. That is, $\sup_{\nu \geqslant 0} \|x_{i+\nu} - x_i\| \to 0$ as $i \to \infty$.

(g) $\{x_i; i \geqslant 0\}$ has *null increment sequences*. That is, $\|x_{i+\nu} - x_i\| \to 0$ as $i \to \infty$, for each $\nu \geqslant 0$.

We note that the diagram

$$\begin{array}{ccccccccccccc}
 & & (c) & & \overset{\Rightarrow}{\not\Leftarrow} & & (b) & & & & (a) & & \\
 & \nearrow & & \not\searrow & & \not\swarrow & & \searrow & & \nearrow & & \not\searrow\ \not\nwarrow & \\
(d) & & \Rightarrow & & (e) & & \Rightarrow & & (f) & & \Rightarrow & & (g)
\end{array}$$

holds true, which completely characterizes every possible single relationship between each of the above assertions. Moreover

$$(e,\ b) \Rightarrow (d),$$

and

$$(g,\ a) \not\Rightarrow (f).$$

However, if X is finite dimensional and (a) holds true, then (g)$\Leftrightarrow$(f) (cf. [21, pp. 55–58] for a proof in $\mathbb{R}^1$, whose extention to $\mathbb{F}^n$ is straightforward by the Bolzano–Weierstrass theorem). The only nontrivial results in the above diagram are (c) $\not\Rightarrow$ (e) and (g) $\not\Rightarrow$ (a), which are readily verified by taking real-valued sequences $\{\xi_i;\ i \geqslant 0\} = \{1, 0, 1/2^2,\ 0, 0, 1/3^2,\ 0,\ 0,\ 0,\ 1/4^2, \ldots\}$ and $\{\xi_i = lg(i+1);\ i \geqslant 0\}$, respectively. It is a simple matter to show that (e, b)$\Rightarrow$(e, c)$\Rightarrow$(d) by using the triangle inequality. It can also be shown that (g, a) $\not\Rightarrow$ (f) by taking the following l_1-valued sequence: $\{x_i = z_i - y_i;\ i \geqslant 0\}$ with $z_i = \{\zeta_j; j \geqslant 1\} \in l_1$, where $\zeta_j = j^{-1}$ if $1 \leqslant j \leqslant i+1$ and $\zeta_j = 0$ if $j > i+1$, and $y_i = \{[i/(i+1)]^j/j;\ j \geqslant 1\} \in l_1$, for each $i \geqslant 0$ (cf. [22, p.78]).

PROPOSITION (P-2). *Let $\{x_i;\ i \geqslant 0\}$ be a sequence in a normed linear space X. If there exist $a = (\alpha_0, \alpha_1, ...) \in l_1$ and $c = (\gamma_0, \gamma_1, ...) \in l_1$ such that*

$$\| x_{i+v} - x_i \| \leqslant |\alpha_i| \, \| x_v - x_0 \| + |\gamma_i|, \qquad \forall i, v \geqslant 0,$$

then $\{x_i; i \geqslant 0\}$ is a Cauchy summable sequence.

Proof. By the triangle inequality, and setting $v = 1$, one has

$$\| x_{i+1} \| \leqslant \| x_{i+1} - x_i \| + \| x_i \| \leqslant \| x_i \| + |\alpha_i| \, \| x_1 - x_0 \| + |\gamma_i|$$

for every $i \geqslant 0$. On iterating the above inequality from $\| x_0 \|$ onwards one gets by induction (cf. Proposition (P-1))

$$\| x_i \| \leqslant \| x_0 \| + \sum_{j=0}^{i-1} (|\alpha_j| \, \| x_1 - x_0 \| + |\gamma_j|), \qquad \forall i \geqslant 1.$$

Hence $\sup_{i \geqslant 0} \| x_i \| \leqslant \| x_0 \| + \| x_1 - x_0 \| \, \| a \|_{l_1} + \| c \|_{l_1} < \infty$. Then,

$$\sum_{i=0}^{\infty} \sup_{v \geqslant 0} \| x_{i+v} - x_i \| \leqslant (\sup_{v \geqslant 0} \| x_v - x_0 \|) \, \| a \|_{l_1} + \| c \|_{l_1} < \infty. \quad \blacksquare$$

LEMMA (L-1). *Let $\{v_i; i \geqslant 0\}$ be any Cauchy summable sequence in a normed linear space X, and consider an X-valued sequence $\{x_i; i \geqslant 0\}$ as follows.*

$$x_{i+1} = Lx_i + Tx_i + v_i, \qquad x_0 \in X \textit{ arbitrary},$$

where $L \in \mathscr{B}[X]$ is uniformly asymptotically stable, and $T: X \to X$ is a contraction. That is, there exist real constants $0 < \alpha < 1$, $\sigma \geqslant 1$, and $0 \leqslant \mu < 1$ such that

(i) $\| L^i \| \leqslant \sigma \alpha^i, \qquad \forall i \geqslant 0,$

(ii) $\| Tx - Ty \| \leqslant \mu \, \| x - y \|, \qquad \forall x, y \in X.$

If

(iii) $\alpha + \sigma\mu < 1,$

then $\{x_i;\ i \geqslant 0\}$ is a Cauchy summable sequence. Moreover if X is a Banach space, then $x = \lim_{i \to \infty} x_i$ is the only solution of

(iv) $y = Ly + Ty + v,$

where $v = \lim_{i \to \infty} v_i$. The solution does not depend on the initial condition x_0.

Proof. (I) It is readily verified by induction that

$$x_{i+v} = L^i x_v + \sum_{j=0}^{i-1} L^{i-j-1}(Tx_{j+v} + v_{j+v})$$

for every $v \geqslant 0$ and $i \geqslant 1$. Hence

$$x_{i+v} - x_i = L^i(x_v - x_0) + \sum_{j=0}^{i-1} L^{i-j-1}(Tx_{j+v} - Tx_j + v_{j+v} - v_j).$$

Therefore, by (i) and (ii), and according to (P-1),

$$\|x_{i+v} - x_i\| \leqslant \sigma \left[\alpha^i \|x_v - x_0\| + \sum_{j=0}^{i-1} \alpha^{i-j-1}(\mu \|x_{j+v} - x_j\| + \|v_{j+v} - v_j\|) \right],$$

$$\|x_{i+v} - x_i\| \leqslant \sigma \left[(\alpha + \sigma\mu)^i \|x_v - x_0\| + \sum_{j=0}^{i-1} (\alpha + \sigma\mu)^{i-j-1} \|v_{j+v} - v_j\| \right]$$

for every $v \geqslant 0$ and $i \geqslant 1$. Hence, since $\sigma \geqslant 1$ and $(\alpha + \sigma\mu) > 0$,

$$\|x_{i+v} - x_i\| \leqslant \alpha_i \|x_v - x_0\| + \gamma_i$$

for every i, $v \geqslant 0$, where

$$\alpha_i = \sigma(\alpha + \sigma\mu)^i,$$

$$\beta_i = (\alpha + \sigma\mu)^{-1} \sup_{v \geqslant 0} \|v_{i+v} - v_i\|,$$

$$\gamma_i = \sum_{j=0}^{i} \alpha_{i-j}\beta_j = \sigma \sum_{j=0}^{i} (\alpha + \sigma\mu)^{i-j-1} \sup_{v \geqslant 0} \|v_{j+v} - v_j\|,$$

for each $i \geqslant 0$. From (iii), and recalling that $\{v_i;\ i \geqslant 0\}$ is Cauchy summable, it follows that $a = (\alpha_0, \alpha_1, ...) \in l_1$, and $b = (\beta_0, \beta_1, ...) \in l_1$. Thus the convolution (or the Cauchy product) of a and b lies in l_1 as well (cf. [22, p. 77]). That is, $c = a * b = (\gamma_0, \gamma_1, ...) \in l_1$. Then $\{x_i;\ i \geqslant 0\}$ is Cauchy summable by (P-2).

(II) Of course the limits v, $x \in X$ do exist, since $\{v_i;\ i \geqslant 0\}$ and $\{x_i;\ i \geqslant 1\}$ are Cauchy sequences (cf. Remark (R-2)) and X is a Banach space. It is immediately verified that the limit $x \in X$ is actually a solution of (iv), since L and T are continuous operators. On the other hand, if $y \in X$ is any solution of (iv), we get

$$x_{i+1} - y = L(x_i - y) + Tx_i - Ty + v_i - v, \qquad \forall i \geqslant 0.$$

On iterating the above equation from x_0 onwards, and then proceeding exactly as in part (I), it follows that

$$\|x_i - y\| \leqslant \alpha_i \|x_0 - y\| + \gamma'_i, \qquad \forall i \geqslant 0,$$

where $c' = (\gamma'_0, \gamma'_1, ...) = a * b'$, and $b' = (\beta'_0, \beta'_1, ...)$ with $\beta'_i = (\alpha + \sigma\mu)^{-1} \|v_i - v\|$ for every $i \geqslant 0$. Now, since $\{v_i - v;\ i \geqslant 0\}$ is a Cauchy summable null sequence, it follows that it is absolutely summable (cf. Remark (R-2), (e, b) $\Rightarrow$ (c)). Therefore $b' \in l_1$. Hence $c' \in l_1$, since $a \in l_1$. Then $\{x_i - y; i \geqslant 0\}$ is absolutelly summable and so it is a null sequence. Thus $y = x$ by uniqueness of the limit x. Since (iv) has a unique solution it has to be independent of x_0. ∎

Remarks

Consider the discrete-time dynamical system in (L-1), and note that L is not necessarily a contraction (thus the possibly nonlinear operator $(T+L)$ may not be a contraction as well).

Remark (R-3). It can be shown that condition (iii) can be replaced by

$$\sigma\alpha + \mu < 1.$$

However, such an assumption is stronger than (iii) (i.e., it implies (iii)). Indeed, if the above holds it necessarily follows by (i) that $\|L\| \leqslant \sigma\alpha < 1$ (i.e., L is a contraction), which implies that we can choose $\sigma = 1$ and $\alpha = \|L\|$ (since $\|L^i\| \leqslant \|L\|^i$; $\forall i \geqslant 0$), thus (iii) holds with $\sigma = 1$. Acutally, if $T \in \mathscr{B}[X]$, then $\|L + T\| \leqslant \|L\| + \|T\| \leqslant \sigma\alpha + \mu$.

Remark (R-4). Obviously if $T \in \mathscr{B}[X]$, then conditions (i), (ii), and (iii) can be replaced by

$$\|(L+T)^i\| \to 0 \qquad \text{as} \quad i \to \infty,$$

or equivalently in a Banach space setting (cf. Remark (R-1)),

$$r_\sigma(L+T) < 1.$$

However, condition (iii), while seemingly stringent when $T \in \mathscr{B}[X]$, is often satisfied in practical cases where the analysis of $\|(L+T)^i\|$ or $r_\sigma(L+T)$ is a puzzling task. For instance, the preceding lemma will play a major role for the stability theorems of Section 4, where X will be a noncommutative Banach algebra, and $(L+T) \in \mathscr{B}[X]$ will be a symmetric linear combination with X-valued coefficients. For example,

$$(L+T)[Q] = \sum_{k=0}^{p} F_k Q F_k^*, \qquad \forall Q \in X,$$

with $F_k \in X$ for each $k = 0, 1, \ldots, p$. In such a case both $\|(L+T)^i\|$ and $r_\sigma(L+T)$ have not generally a simple form in terms of the X-valued coefficients $\{F_k; k = 0, 1, \ldots, p\}$, unless these coefficients commute.

Remark (R-5). Finally note that if X is a Banach space and $L, T \in \mathscr{B}[X]$ commute, then

$$r_\sigma(L+T) \leqslant r_\sigma(L) + r_\sigma(T) \leqslant \alpha + \mu,$$

since the first inequality holds whenever L and T commute (cf. [23, p. 45]), and $r_\sigma(L) \leqslant \alpha$ (cf. Remark (R-1)), $r_\sigma(T) \leqslant \|T\| \leqslant \mu$. Therefore the coefficent $\sigma \geqslant 1$ in (iii) can be thought of as a cost to be paid when L and T do not commute.

3. Bilinear System Correlation Evolution

Model Description

Consider a discrete-time dynamical system operating in a stochastic environment, whose evolution is governed by the following finite-dimensional difference equation

$$x(i+1) = \left[A_0 + \sum_{k=1}^{\mathrm{p}} A_k \omega_k(i)\right] x(i) + Bu(i), \qquad x(0) = x_0, \tag{1}$$

where $A_k \in \mathscr{B}[\mathbb{F}^n]$, for each $k = 0, 1, \ldots, \mathrm{p}$, and $B \in \mathscr{B}[\mathbb{F}^m, \mathbb{F}^n]$. Here $\{u(i); i \geqslant 0\}$ and $\{x(i); i \geqslant 0\}$ are random sequences in $\mathbb{F}^m$ and $\mathbb{F}^n$, denoting input disturbance and model state, respectively. The $\mathbb{F}^{\mathrm{p}}$-valued random sequence $\{w(i) = (\omega_1(i), \ldots, \omega_{\mathrm{p}}(i)); i \geqslant 0\}$ can be viewed either as an input disturbance which may eventually be equal to $\{u(i); i \geqslant 0\}$, or as an internal model disturbance. In the former case the $\mathscr{B}[\mathbb{F}^n]$-valued random sequence $\{W(i); i \geqslant 0\}$, given by

$$W(i) = \sum_{k=1}^{\mathrm{p}} A_k \omega_k(i),$$

represents the multiplicative action of an input disturbance on the model state, and (1) is said to be a time-invariant bilinear model. In the latter case (1) is said to be a linear stochastic model, with $[A_0 + W(i)]$ denoting the random system operator, whenever $\{w(i); i \geqslant 0\}$ and $\{u(i); i \geqslant 0\}$ are mutually independent.

Assumptions on the Stochastic Environment

Throughout this paper the symbol E will stand for expectation as usual. The stochastic environment, under which the model (1) operates, is characterized by the initial condition x_0 and the random sequences $\{w(i); i \geqslant 0\}$ and $\{u(i); i \geqslant 0\}$. We assume that:

(A-1) x_0 is a second order random vector independent of $\{w(i), u(i); i \geqslant 0\}$.

(A-2) $\{w(i); i \geqslant 0\}$ and $\{u(i); i \geqslant 0\}$ are both independent second order wide sense stationary random sequences.

(A-3) $\{w(j), u(j)\}$ is independent of $\{w(i), u(i); j \neq i \geqslant 0\}$ for every $j \geqslant 0$, and $E\{w(i)\, u^*(i)\}$ is constant.

It is worth noting that assumptions (A-2) and (A-3) are both satisfied if $\{(w(i), u(i)); i \geqslant 0\}$ is an independent second order wide sense stationary random sequence in $\mathbb{F}^{p+m}$. Moreover, if assumption (A-2) holds true, and if $\{w(i); i \geqslant 0\}$ and $\{u(i); i \geqslant 0\}$ are either related by means of a linear transformation or mutually independent, then assumption (A-3) is satisfied.

The Independence Argument

Independence in (A-1) to (A-3) can be relaxed, such that the followng uncorrelatedness-type assumption is sufficient. For each $0 \leqslant j \leqslant i$,

$$E\{w(i)\, x^*(i)\} = E\{w(i)\}\, E\{x(i)\}^*,$$
$$E\{u(i)\, x^*(j)\} = E\{u(i)\}\, E\{x(j)\}^*,$$
$$E\{W(i)\, x(i)\, x^*(j)\} = E\{W(i)\}\, E\{x(i)\, x^*(j)\},$$
$$E\{W(i)\, x(i)\, u^*(i)\} = E\{W(i)\, E\{x(i)\}\, u^*(i)\},$$
$$E\{W(i)\, x(i)\, x^*(i)\, W^*(i)\} = E\{W(i)\, E\{x(i)\, x^*(i)\}\, W^*(i)\},$$

where $x(i) = \Phi_w(i, 0)\, x_0 + \sum_{j=0}^{i-1} \Phi_w(i, j+1)\ Bu(j)$ for every $i \geqslant 1$, with $\Phi_w(i, i) = I$ and $\Phi_w(i, j) = [A_0 + W(i-1)] \cdots [A_0 + W(j)]$ for $i > j$. It is clear that independence, as supposed in (A-1) to (A-3), implies the above somewhat artificial uncorrelatedness-type assumption.

Auxiliary Notation

We summarize here all the auxiliary notation that will be required in the sequel. Regarding the second order jointly wide sense stationary disturbances $\{w(i); i \geqslant 0\}$ and $\{u(i); i \geqslant 0\}$, set

$$\rho_k = E\{\omega_k(i)\},$$
$$\rho_{kl} = E\{\omega_k(i)\, \overline{\omega_l(i)}\},$$
$$\gamma_{kl} = \rho_{kl} - \rho_k \overline{\rho_l},$$

in $\mathbb{F}$ for each k, $l = 1, \ldots, \mathfrak{p}$, and

$$r = E\{u(i)\},$$
$$r_k = E\{\overline{\omega_k(i)}\, u(i)\},$$
$$R = E\{u(i)\, u^*(i)\},$$
$$C = E\{w(i)\, w^*(i)\} - E\{w(i)\}\, E\{w(i)\}^* = [\gamma_{kl}],$$

in $\mathbb{F}^m$, $\mathbb{F}^m$, $\mathscr{B}[\mathbb{F}^m]^+$, and $\mathscr{B}[\mathbb{F}^{\mathrm{p}}]^+$, respectively, for every $i \geqslant 0$. Moreover set

$$M = E\{W(i)\} = \sum_{k=1}^{\mathrm{p}} \rho_k A_k,$$
$$F = F_0 = A_0 + M,$$
$$F_k = \gamma_{kk}^{1/2} A_k,$$

for each $k = 1, \ldots, \mathrm{p}$, in $\mathscr{B}[\mathbb{F}^n]$, and define operators T and T_l in $\mathscr{B}[\mathscr{B}[\mathbb{F}^n]]$ as follows. For any $Q \in \mathscr{B}[\mathbb{F}^n]$,

$$T(Q) = E\{[W(i) - M]\, Q[W(i) - M]^*\} = \sum_{k,l=1}^{\mathrm{p}} \gamma_{kl} A_k Q A_l^*,$$

$$T_l(Q) = \sum_{\substack{k=0 \\ k \neq l}}^{\mathrm{p}} F_k Q F_k^*$$

for each $l = 0, 1, \ldots, \mathrm{p}$. Considering the state sequence generated by (1), set

$$q(i) = E\{x(i)\}$$

in $\mathbb{F}^n$, and for each $v \geqslant 0$

$$Q_v(i) = E\{x(i+v)\, x^*(i)\},$$
$$Q(i) = Q_0(i),$$

in $\mathscr{B}[\mathbb{F}^n]$, and $\mathscr{B}[\mathbb{F}^n]^+$, for every $i \geqslant 0$. Note that the existence of $q(i)$ and $Q_v(i)$, for each $i \geqslant 0$, is guaranteed by independence and second order assumptions, thus $\{x(i);\, i \geqslant 0\}$ is a second order random sequence as well. Finally, regarding the mixed terms involving both disturbances and state statistics set, in $\mathscr{B}[\mathbb{F}^n]$ for each $i \geqslant 0$,

$$P(i) = \left[A_0 q(i)\, r^* + \sum_{k=1}^{\mathrm{p}} A_k q(i)\, r_k^* \right] B^*,$$
$$V(i) = P(i) + P^*(i) + BRB^*.$$

Mean and Correlation Evolution of the State Sequence

Given a discrete dynamical system as in (1), and the preceding assumptions on the stochastic environment, we now seek for the evolution equations of the state mean $q(i)$ and correlations $Q_\nu(i)$. By (1), and the above independence argument, it is immediate to show that

$$q(i+1) = Fq(i) + Br, \qquad \forall i \geqslant 0, \tag{2}$$

such that, by induction,

$$q(i) = F^i q(0) + \sum_{j=0}^{i-1} F^j Br, \qquad \forall i \geqslant 1.$$

It is also readily verified by the independence argument that $P(i) = E\{[A_0 + W(i)]\, x(i)\, u^*(i)\, B^*\}$. Moreover, since $FQF^* + T(Q) = E\{[A_0 + W(i)]\, Q[A_0 + W(i)]^*\}$ for any $Q \in \mathscr{B}[\mathbb{F}^n]$, we get $FQ(i)\, F^* + T[Q(i)] = E\{[A_0 + W(i)]\, x(i)\, x^*(i)[A_0 + W(i)]^*\}$ by the independence argument. Thus following from (1)

$$Q(i+1) = FQ(i)\, F^* + T[Q(i)] + V(i), \qquad \forall i \geqslant 0. \tag{3a}$$

If the model disturbance covariance matrix C is diagonal (i.e., if the disturbance $w(i)$ has uncorrelated components for every $i \geqslant 0$, such that $\gamma_{kl} = 0$ whenever $k \neq l$), then (3a) becomes

$$Q(i+1) = F_l Q(i)\, F_l^* + T_l[Q(i)] + V(i), \qquad \forall i \geqslant 0, \tag{3b}$$

for each $l = 0, 1, ..., \mathfrak{p}$, or equivalently,

$$Q(i+1) = \sum_{k=0}^{\mathfrak{p}} F_k Q(i)\, F_k^* + V(i), \qquad \forall i \geqslant 0, \tag{3c}$$

whose solution is readily obtained by induction:

$$Q(i) = L^i[Q(0)] + \sum_{j=0}^{i-1} L^{i-j-1}[V(j)], \qquad \forall i \geqslant 1,$$

with $L[Q] = \sum_{k=0}^{\mathfrak{p}} F_k Q F_k^*$ for all $Q \in \mathscr{B}[\mathbb{F}^n]$, such that $\{L^i \in \mathscr{B}[\mathscr{B}[\mathbb{F}^n]];\ i \geqslant 1\}$ is given by

$$L^i(Q) = \sum_{k_i=0}^{\mathfrak{p}} \cdots \sum_{k_1=0}^{\mathfrak{p}} F_{k_i} \cdots F_{k_1} Q F_{k_1}^* \cdots F_{k_i}^*, \qquad \forall Q \in \mathscr{B}[\mathbb{F}^n].$$

Similarly it can also be shown from (1) that

$$Q_{\nu+1}(i) = FQ_\nu(i) + Brq^*(i), \qquad \forall \nu, i \geqslant 0.$$

On iterating the above equation from $Q_0(i) = Q(i)$ onwards we get by induction in $v \geqslant 1$,

$$Q_v(i) = F^v Q(i) + \sum_{j=0}^{v-1} F^j Brq^*(i), \qquad \forall i \geqslant 0. \tag{4}$$

Remark (R-6). By definition, T is $\mathscr{B}[\mathbb{F}^n]^+$-invariant, so that $T[Q(i)] \in \mathscr{B}[\mathbb{F}^n]^+$ for every $i \geqslant 0$.[1] On the other hand, as it also happens in the linear case, $\{Q(i) \in \mathscr{B}[\mathbb{F}^n]^+;\ i \geqslant 0\}$ is not necessarily a monotone sequence. That is, it may happen that $\pm[Q(i+1) - Q(i)] \notin \mathscr{B}[\mathbb{F}^n]^+$. Moreover it can also be verified that, for a given $i \geqslant 0$, $V(i) \notin \mathscr{B}[\mathbb{F}^n]^+$ in general (thus implying that $P(i) + P^*(i)$ does not necessarily lie in $\mathscr{B}[\mathbb{F}^n]^+$), even though all other terms in (3) lie in $\mathscr{B}[\mathbb{F}^n]^+$.

4. Mean Square Stability

Consider the mean $\{q(i);\ i \geqslant 0\}$ and correlation $\{Q(i);\ i \geqslant 0\}$ sequences given in (2) and (3). In this section we investigate sufficient conditions on the stochastic model $([A_0 + \sum_{k=1}^{p} A_k \omega_k(\mathrm{i})], B)$ described in (1), in order to ensure that $\{q(i) = E\{x(i)\};\ i \geqslant 0\}$ and $\{Q(i) = E\{x(i)\, x^*(i)\};\ i \geqslant 0\}$ converge for any admissible initial condition x_0 and input disturbance $\{u(i);\ i \geqslant 0\}$, and their limits do not depend on x_0.

Definition (D-1). The model in (1) is *mean square stable* (m.s.s.) if, for any initial condition x_0 and input disturbance $\{u(i);\ i \geqslant 0\}$ satisfying assumptions (A-1)–(A-3), there exist $q \in \mathbb{F}^n$ and $Q \in \mathscr{B}[\mathbb{F}^n]^+$ independent of x_0 such that

(a) $\|q(i) - q\| \to 0$ as $i \to \infty$,

(b) $\|Q(i) - Q\| \to 0$ as $i \to \infty$.

Remark (R-7). The second order state sequence $\{x(i);\ i \geqslant 0\}$ is said to be *asymptotically wide sense stationary* (a.w.s.s.) if there exist $q \in \mathbb{F}^n$ and $Q_v \in \mathscr{B}[\mathbb{F}^n]$, for each $v \geqslant 0$, such that

$$\|q(i) - q\| \to 0 \qquad \text{as} \quad i \to \infty,$$

$$\|Q_v(i) - Q_v\| \to 0 \qquad \text{as} \quad i \to \infty.$$

Note that (1) is m.s.s. if and only if $\{x(i);\ i \geqslant 0\}$ is a.w.s.s. for any x_0 and $\{u(i);\ i \geqslant 0\}$ as in (A-1)–(A-3), and the limits q and Q_v do not depend on

[1] "En passant," this shows that the assumption (A-3,b) in [19] may be dropped out.

x_0. Also note that, if there exists the limit $Q_0 = Q \in \mathscr{B}[\mathbb{F}^n]$ it must be in $\mathscr{B}[\mathbb{F}^n]^+$, since $Q(i) \in \mathscr{B}[\mathbb{F}^n]^+$ for every $i \geqslant 0$ and $\mathscr{B}[\mathbb{F}^n]^+$ is closed in $\mathscr{B}[\mathbb{F}^n]$.

THEOREM (T-1). *Consider the evolution equations in* (2) *and* (3a). *Set*

$$\mu^2 = \| T \| \leqslant \sum_{k,l=1}^{p} |\gamma_{kl}| \, \| A_k \| \, \| A_l \|.$$

Suppose there exist real constants $\sigma \geqslant 1$ *and* $0 < \alpha < 1$ *such that*

$$\| F^i \| \leqslant \sigma \alpha^i, \qquad \forall i \geqslant 0.$$

If

$$\alpha^2 + \sigma^2 \mu^2 < 1, \tag{5}$$

then the model in (1) *is* m.s.s.

Proof. (a) Consider Eq. (2). From Lemma (L-1) (with $X = \mathbb{F}^n$, $T = 0$, $L = F$, and $\{v_i = Br;\ i \geqslant 0\}$) it follows that $\{q(i);\ i \geqslant 0\}$ is Cauchy summable, and so (D-1, a) holds.

(b) Now consider Lemma (L-1) in the Banach space $X = \mathscr{B}[\mathbb{F}^n]$ with $T \in \mathscr{B}[\mathscr{B}[\mathbb{F}^n]]$ and $\{V(i) \in \mathscr{B}[\mathbb{F}^n];\ i \geqslant 0\}$ as defined in Section 3, and let $L \in \mathscr{B}[\mathscr{B}[\mathbb{F}^n]]$ be given by

$$L(Q) = FQF^*, \qquad \forall Q \in \mathscr{B}[\mathbb{F}^n],$$

such that

$$\| L^i \| = \sup_{\| Q \| = 1} \| F^i Q F^{*i} \| = \| F^i \|^2 \leqslant \sigma^2 \alpha^{2i}, \qquad \forall i \geqslant 0.$$

Form part (a) it follows that $\{V(i);\ i \geqslant 0\}$ is Cauchy summable since, for every $i, v \geqslant 0$,

$$\begin{aligned}
\| V(i+v) - V(i) \| &= \| P(i+v) - P(i) + P^*(i+v) - P^*(i) \| \\
&\leqslant 2 \, \| P(i+v) - P(i) \| \\
&\leqslant 2\left(\| A_0 \| \, \| r \| + \sum_{k=1}^{p} \| A_k \| \, \| r_k \| \right) \| B \| \, \| q(i+v) - q(i) \|.
\end{aligned}$$

Then, if (5) holds we get the result in (D-1, b), with $Q \in \mathscr{B}[\mathbb{F}^n]^+$ (cf. Remark (R-7)), according to Eq. (3a) and Lemma (L-1). ∎

Remarks

Consider the assumptions of (T-1). Since $F = A_0 + M$ is (uniformly) asymptotically stable (cf. Remark (R-1)) it follows that (cf. [22, p. 567]) $\sum_{j=0}^{\infty} F^j = (I-F)^{-1} \in \mathscr{B}[\mathbb{F}^n]$. Hence

$$q = (I-F)^{-1} Br,$$

where $q = \lim_{i\to\infty} q(i)$ according to (2). Moreover, since $V(i) \to V$ as $i \to \infty$, with

$$V = P + P^* + BRB^*,$$

$$P = A_0(I-F)^{-1} Brr^*B^* + \sum_{k=1}^{p} A_k(I-F)^{-1} Br\, r_k^* B^*,$$

it is readily verified from (3a), (L-1), and (R-7) that $Q = \lim_{i\to\infty} Q(i) \in \mathscr{B}[\mathbb{F}^n]^+$ is the only solution of

$$Q = FQF^* + T(Q) + V.$$

Moreover, with $Q_\nu = \lim_{i\to\infty} Q_\nu(i)$ for each $\nu \geqslant 0$, it follows that $qq^* = \lim_{\nu\to\infty} Q_\nu$, since by (4)

$$Q_\nu = F^\nu Q + \sum_{j=0}^{\nu-1} F^j Brr^*B^*(I-F^*)^{-1}, \qquad \forall \nu \geqslant 1.$$

In the following we make several comments on condition (5), as a continuation of those considered in (R-3)–(R-5). Some of these will motivate the next theorems.

Remark (R-8). A condition somewhat similar to (5) was given in [18], for stability with probability one (w.p.1) of free stochastic systems under ergodicity assumption, as follows: Set $Bu = 0$ in (1) and assume, instead of (A-2), that

$$\frac{1}{i} \sum_{j=0}^{i-1} \| W(j) \| \xrightarrow{\text{w.p.1}} E\{\| W(0) \|\} \qquad \text{as} \qquad i \to \infty.$$

Let A_0 be asymptotically stable, that is (cf. Remark (R-1))

$$\| A_0^i \| \leqslant \sigma\alpha^i, \qquad \forall i \geqslant 0,$$

with $\sigma \geqslant 1$ and $0 < \alpha < 1$. If

$$\alpha + \sigma E\{\| W(0) \|\} < 1,$$

then the free model (i.e., with $Bu=0$) in (1) is asymptotically stable with probability one. That is, for any bounded (w.p.1) initial condition,

$$x(i) \xrightarrow{\text{w.p.1}} 0 \quad \text{as} \quad i \to \infty.$$

Remark (R-9). Condition (5), which is sufficient to ensure mean square stability in (T-1), is not a necessary one. To illustrate this, set $n=2$, $m=p=1$, $\gamma_{11}=R=1$, $\rho_1=r=0$, $q(0)=0$ (such that $q(i)=0$ and $P(i)=M=0$ for every $i \geqslant 0$), and

$$A_0=\begin{bmatrix} 0 & \alpha_1 \\ \alpha_2 & 0 \end{bmatrix}, \quad A_1=\begin{bmatrix} 0 & \beta_1 \\ \beta_2 & 0 \end{bmatrix}, \quad B=b=\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad Q(0)=\operatorname{diag}(\eta_1(0), \eta_2(0)),$$

where α_1, α_2, β_1, and β_2 are nonzero real constants. Therefore

$$\|F^i\| = \|A_0^i\| = \begin{cases} \alpha^i, & i=0, 2, 4,\ldots \\ \sigma\alpha^i, & i=1, 3, 5,\ldots \end{cases}$$

$$\alpha = r_\sigma(A_0) = |\alpha_1\alpha_2|^{1/2},$$

$$\sigma^2 = \frac{\|A_0\|^2}{r_\sigma(A_0)^2} = \frac{\max\{|\alpha_1|, |\alpha_2|\}}{\min\{|\alpha_1|, |\alpha_2|\}},$$

$$\mu = \|A_1\| = \max\{|\beta_1|, |\beta_2|\}.$$

From (3a) it is readily verified by induction that, for every $i \geqslant 0$, $Q(i)=\operatorname{diag}(\eta_1(i), \eta_2(i))$, where $h(i)=(\eta_1(i), \eta_2(i))$ in $\mathbb{R}^2$ is given by

$$h(i+1)=Ah(i)+b, \qquad h(0)=(\eta_1(0), \eta_2(0)),$$

with

$$A=\begin{bmatrix} 0 & \alpha_1^2+\beta_1^2 \\ \alpha_2^2+\beta_2^2 & 0 \end{bmatrix},$$

such that

$$r_\sigma(A)^2=(\alpha_1^2+\beta_1^2)(\alpha_2^2+\beta_2^2)=r_\sigma(A_0^2+A_1^2)^2+r_\sigma([A_0, A_1])^2,$$

where $[A_0, A_1]=A_0A_1-A_1A_0$. After some algebraic manipulation it can be shown that $r_\sigma(A) \leqslant \alpha^2+\sigma^2\mu^2$. Actually, if either $|\alpha_1| \neq |\alpha_2|$ or $|\beta_1| \neq |\beta_2|$ it follows that

$$r_\sigma(A) < \alpha^2+\sigma^2\mu^2.$$

Hence it may happen that

$$r_\sigma(A)<1 \qquad \text{and} \qquad \alpha^2+\sigma^2\mu^2>1$$

(e.g., set $\alpha_1 = 1$, $\alpha_2 = \frac{1}{4}$, $\beta_1 = \beta_2 = \frac{1}{2}$, such that $r_\sigma(A) = \frac{5}{8}$ and $\alpha^2 + \sigma^2\mu^2 = \frac{5}{4}$). However, as it is well known, $\{h(i);\ i \geq 0\}$ converges (or, equivalently, $\{Q(i);\ i \geq 0\}$ converges) for any $h(0)$ to a limit independent of $h(0)$ whenever $r_\sigma(A) < 1$, although condition (5) may not be satisfied when $r_\sigma(A) < 1$. Moreover, if we replace the conditions imposed on the matrix A_0 by the conditions imposed on the matrix A_1, and vice versa, we get $r_\sigma(A) \leq \underline{\alpha}^2 + \underline{\sigma}^2\underline{\mu}^2$, where

$$\| A_1^i \| = \begin{cases} \underline{\alpha}^i, & i = 0, 2, 4, \ldots \\ \underline{\sigma\alpha}^i, & i = 1, 3, 5, \ldots \end{cases}$$

$$\underline{\alpha} = r_\sigma(A_1) = |\beta_1\beta_2|^{1/2},$$

$$\underline{\sigma}^2 = \frac{\| A_1 \|^2}{r_\sigma(A_1)^2} = \frac{\max\{|\beta_1|, |\beta_2|\}}{\min\{|\beta_1|, |\beta_2|\}},$$

$$\underline{\mu} = \| A_0 \| = \max\{|\alpha_1|, |\alpha_2|\},$$

since this is equivalent to replacing the matrix A_0 by the matrix A_1 (which have the same structure). Therefore, for this particular case, we have another sufficient condition for mean square stability, that is,

$$\underline{\alpha}^2 + \underline{\sigma}^2\underline{\mu}^2 < 1,$$

which is not implied by condition (5). For instance, set $\beta_1 = 1$, $\beta_2 = \beta$ for any $0 < \beta < 1$, and $\alpha_1 = \alpha_2 = \alpha$ for any $0 < \alpha^2 < \beta(1 - \beta) < 1$. Then (5) is not satisfied since

$$\alpha^2 + \sigma^2\mu^2 = \alpha^2 + 1 > 1,$$

although mean square stability is guaranteed by the new condition

$$r_\sigma(A) < \underline{\alpha}^2 + \underline{\sigma}^2\underline{\mu}^2 = \beta + \alpha^2/\beta < 1.$$

This particular case will be generized in Theorem (T-2) whenever the model disturbance covariance matrix C is diagonal.

Remark (R-10). On the other hand, the weaker condition

$$\alpha^2 + \mu^2 < 1$$

is not sufficient to ensure mean square stability, unless $\{A_k;\ k = 0, 1, \ldots, \mathfrak{p}\}$ commute (cf. Remark (R-5)). For instance, regarding the model described in the preceding remark, set $Q(0) = I$, $\alpha_1 = \alpha^{-2}$, $\alpha_2 = \alpha^4$, and $\beta_1 = \beta_2 = \alpha$, for any $0 < \alpha < 1$, such that

$$r_\sigma(A)^2 = \alpha^{10} + 2\alpha^4 + \alpha^{-2} > 1, \qquad \forall \alpha \in (0, 1).$$

Thus $\| Q(i) \| \to \infty$ as $i \to \infty$ for every $\alpha \in (0, 1)$, although

$$\alpha^2 + \mu^2 = 2\alpha^2 < 1, \qquad \forall \alpha \in (0, 1/\sqrt{2}).$$

Still regarding the example discussed in the preceding remarks we finally note that, if A_0 and A_1 commute (i.e., if $\alpha_1 \beta_2 = \alpha_2 \beta_1 = \delta$), then the weakest condition $r_\sigma(A_0^2 + A_1^2) < 1$, which does not involve either the norm $\mu = \| A_1 \|$ or $\underline{\mu} = \| A_0 \|$, is sufficient to ensure mean square stability. Actually, if A_0 and A_1 commute (or equivalently, if $[A_0, A_1] = 0$), we have in this particular case

$$\begin{aligned} r_\sigma(A)^2 &= r_\sigma(A_0^2 + A_1^2)^2 = (r_\sigma(A_0)^2 + r_\sigma(A_1)^2)^2 \\ &= (\alpha_1 \alpha_2)^2 + (\beta_1 \beta_2)^2 + 2\delta^2 \leqslant (\alpha^2 + \mu^2)^2. \end{aligned}$$

It will be shown in Theorem (T-3) that $r_\sigma(A_0)^2 + r_\sigma(A_1)^2 < 1$ is sufficient to ensure mean square stability in general, whenever A_0 and A_1 commute and C is a diagonal matrix.

THEOREM T-2. *Suppose C is a diagonal matrix, and consider the evolution equations* (2) *and* (3b). *For each $l = 0, 1, ..., \mathrm{p}$ set*

$$\mu_l^2 = \| T_l \| \leqslant \sum_{\substack{k=0 \\ k \neq l}}^{\mathrm{p}} \| F_k \|^2,$$

and let $\sigma_l \geqslant 1$ and $\alpha_l > 0$ be real constants such that

$$\| F_l^i \| \leqslant \sigma_l \alpha_l^i, \qquad \forall i \geqslant 0.$$

If

$$\min_{0 \leqslant l \leqslant \mathrm{p}} (\alpha_l^2 + \sigma_l^2 \mu_l^2) < 1, \tag{6}$$

then the model in (1) *is* m.s.s.

Proof. For any l, $l_0 \in \{0, 1, ..., \mathrm{p}\}$, such that $l \neq l_0$,

$$\begin{aligned} \| F_l \|^2 = \| F_l F_l^* \| &= \sup_{\|x\| = 1} \langle F_l F_l^* x; x \rangle \leqslant \sup_{\|x\| = 1} \left\langle \sum_{\substack{k=0 \\ k \neq l_0}}^{\mathrm{p}} F_k F_k^* x; x \right\rangle \\ &= \left\| \sum_{\substack{k=0 \\ k \neq l_0}}^{\mathrm{p}} F_k F_k^* \right\| \leqslant \sup_{\|Q\| = 1} \left\| \sum_{\substack{k=0 \\ k \neq l_0}}^{\mathrm{p}} F_k Q F_k^* \right\| = \| T_{l_0} \| = \mu_{l_0}^2. \end{aligned}$$

Now, if there exists $l_0 \in \{0, 1, ..., \mathfrak{p}\}$ such that $\alpha_{l_0}^2 + \sigma_{l_0}^2 \mu_{l_0}^2 < 1$, then $\alpha_{l_0} < 1$ and $\mu_{l_0} < 1$ (since $\sigma_{l_0} \geqslant 1$), which implies by the above inequality that $\| F_l \| < 1$ for every $l \neq l_0$. Hence, if (6) holds, with the minimum achieved for some $l_0 \in \{0, 1, ..., \mathfrak{p}\}$, then F_{l_0} is (uniformly) asymptotically stable, and F_l is a contraction for every $l \neq l_0$. In other words, if (6) holds, then the set $\{F_l; l = 0, 1, ..., \mathfrak{p}\}$ is comprised of either $\mathfrak{p}+1$ contractions, or $\mathfrak{p}$ contractions and one (uniformly) asymptotically stable noncontraction. Moreover, in the latter case, the only index for which (6) holds is that associated to the noncontraction matrix. Therefore the desired result follows exactly as in the proof of (T-1), with the evolution Eq. (3a) replaced by (3b), which holds true whenever C is diagonal. ▮

Remark (R-11). Condition (6) in (T-2) can be replaced by

$$\sum_{k=0}^{\mathfrak{p}} \| F_k \|^2 < 1,$$

which obviously implies (6) by setting $\sigma_l = 1$ and $\alpha_l = \| F_l \|$ for each $l = 0, 1, ..., \mathfrak{p}$ (cf. Remark (R-3)).

THEOREM (T-3). *Let C be a diagonal matrix, and consider the evolution equations* (2) *and* (3c). *Suppose the operators $\{F_k; k = 0, 1, ..., \mathfrak{p}\}$ commute. If*

$$\sum_{k=0}^{\mathfrak{p}} r_\sigma(F_k)^2 < 1, \tag{7}$$

then the model in (1) *is* m.s.s.

Proof. First, consider the following auxiliary result.

(AR-1) Let $\mathscr{R}$ be a ring with identity. For any nonnegative integer $\mathfrak{p}$ let $\{a_k; k = 0, 1, ..., \mathfrak{p}\}$ be a commutative set of $\mathscr{R}$. Then, for every $i \in \mathbb{N} = \{0, 1, 2, ...,\}$

$$\left(\sum_{k=0}^{\mathfrak{p}} a_k \right)^i = \sum_{I_i^{\mathfrak{p}}} \frac{i!}{i_0! \cdots i_{\mathfrak{p}}!} \prod_{k=0}^{\mathfrak{p}} a_k^{i_k},$$

where $I_i^{\mathfrak{p}} = \{(i_0, ..., i_{\mathfrak{p}}) \in \mathbb{N}^{\mathfrak{p}+1}: \sum_{k=0}^{\mathfrak{p}} i_k = i\}$. This is the extention of the well-known binominal formula (e.g., see [24, p. 123]), from which the above multinomial one is obtained by induction in $\mathfrak{p}$. We shall be particularly interested in two special cases: $\mathscr{R} = \mathbb{R}$, the real field, and $\mathscr{R} = \mathscr{B}[\mathscr{B}[\mathbb{F}^n]]$.

(a) Consider Eq. (2). Since $r_\sigma(F_0) < 1$ it follows from Remark (R-1) and Lemma (L-1) the result in (D-1, a). Moreover, it also follows that

$\{V(i) \in \mathscr{B}[\mathbb{F}^n];\ i \geqslant 0\}$ as defined in Section 3 is Cauchy summable (cf. proof of (T-1)).

(b) Set $L = \sum_{k=0}^{\mathrm{p}} L_k \in \mathscr{B}[\mathscr{B}[\mathbb{F}^n]]$,
where $\{L_k \in \mathscr{B}[\mathscr{B}[\mathbb{F}^n]];\ k = 0, 1, ..., \mathrm{p}\}$ are defined by the formula

$$L_k(Q) = F_k Q F_k^*, \qquad \forall Q \in \mathscr{B}[\mathbb{F}^n],$$

such that

$$\| L_k^i \| = \sup_{\|Q\| = 1} \| F_k^i Q F_k^{*i} \| = \| F_k^i \|^2, \qquad \forall i \geqslant 0.$$

Note that $\{L_k;\ k = 0, 1, ..., \mathrm{p}\}$ commute by the commutativity assumption on $\{F_k;\ k = 0, 1, ..., \mathrm{p}\}$. Therefore, by (AR-1), we get

$$L^i = \sum_{I_i^{\mathrm{p}}} \frac{i!}{i_0! \cdots i_{\mathrm{p}}!} \prod_{k=0}^{\mathrm{p}} L_k^{i_k}, \qquad \forall i \geqslant 0.$$

Hence

$$\| L^i \| \leqslant \sum_{I_i^{\mathrm{p}}} \frac{i!}{i_0! \cdots i_{\mathrm{p}}!} \prod_{k=0}^{\mathrm{p}} \| F_k^{i_k} \|^2, \qquad \forall i \geqslant 0.$$

Since $r_\sigma(F_k) < 1$ for every $k = 0, 1, ..., \mathrm{p}$, it follows by (R-1) that there exist $\sigma_k \geqslant 1$ and $\varepsilon_k \in (0, 1 - r_\sigma(F_k))$ such that $\| F_k^i \| \leqslant \sigma_k (r_\sigma(F_k) + \varepsilon_k)^i$ for all $i \geqslant 0$. Then, by using (AR-1) again

$$\begin{aligned} \| L^i \| &\leqslant \left(\prod_{k=0}^{\mathrm{p}} \sigma_k^2 \right) \sum_{I_i^{\mathrm{p}}} \frac{i!}{i_0! \cdots i_{\mathrm{p}}!} \prod_{k=0}^{\mathrm{p}} (r_\sigma(F_k) + \varepsilon_k)^{2 i_k} \\ &= \sigma \alpha^i, \qquad \forall i \geqslant 0, \end{aligned}$$

with

$$\sigma = \prod_{k=0}^{\mathrm{p}} \sigma_k^2 \geqslant 1 \qquad \text{and} \qquad \alpha = \sum_{k=0}^{\mathrm{p}} (r_\sigma(F_k) + \varepsilon_k)^2.$$

By (7) and (R-1) we can choose $\{\varepsilon_k;\ k = 0, 1, ..., \mathrm{p}\}$ small, so that $\alpha < 1$. Therefore, since Eq. (3c) can be written as

$$Q(i+1) = L Q(i) + V(i), \qquad \forall i \geqslant 0,$$

and recalling from part (a) that $\{V(i);\ i \geqslant 0\}$ is Cauchy summable, the result in (D-1, b) (with $Q \in \mathscr{B}[\mathbb{F}^n]^+$ according to remark (R-7)) follows from Lemma (L-1) with $T = 0$. ▮

Remark (R-12). Let C be diagonal and $\{A_k;\ k=0, 1,..., \mathrm{p}\}$ be mutually commutative, such that $\{F_k;\ k=0, 1,..., \mathrm{p}\}$ commute. Set

$$|r_w| = (|\rho_1|,..., |\rho_{\mathrm{p}}|) \in \mathbb{R}^{\mathrm{p}},$$

$$|R_w| = [|\rho_{kl}|] \in \mathscr{B}[\mathbb{R}^{\mathrm{p}}]^+,$$

$$S_w = \left[\begin{array}{c|c} 1 & |r_w|^* \\ \hline |r_w| & |R_w| \end{array}\right] \in \mathscr{B}[\mathbb{R}^{\mathrm{p}+1}]^+,$$

$$s_A = (r_\sigma(A_0), r_\sigma(A_1),..., r_\sigma(A_{\mathrm{p}})) \in \mathbb{R}^{\mathrm{p}+1},$$

and note that

$$|\rho_{kl}| = \begin{cases} \rho_{kk} = \gamma_{kk} + |\rho_k|^2, & \text{if} \quad k=l, \\ |\rho_k|\,|\rho_l| \leqslant \rho_{kk}^{1/2}\rho_{ll}^{1/2}, & \text{if} \quad k \neq l, \end{cases}$$

since $C = \mathrm{diag}(\gamma_{11},..., \gamma_{\mathrm{pp}}) \in \mathscr{B}[\mathbb{R}^{\mathrm{p}}]^+$ with $\gamma_{kk} = \rho_{kk} - |\rho_k|^2 \geqslant 0$. We now show that condition (7) in (T-3) can be replaced by

$$\langle S_w s_A ; s_A \rangle < 1.$$

Actually, by the commutativity assumption (cf. [23], p. 45]),

$$\begin{aligned} \sum_{k=0}^{\mathrm{p}} r_\sigma(F_k)^2 &= r_\sigma\left(A_0 + \sum_{k=1}^{\mathrm{p}} \rho_k A_k\right)^2 + \sum_{k=1}^{\mathrm{p}} r_\sigma(\gamma_{kk}^{1/2} A_k)^2 \\ &\leqslant r_\sigma(A_0)^2 + \left(\sum_{k=1}^{\mathrm{p}} |\rho_k|\, r_\sigma(A_k)\right)^2 + 2r_\sigma(A_0) \sum_{k=1}^{\mathrm{p}} |\rho_k|\, r_\sigma(A_k) \\ &\quad + \sum_{k=1}^{\mathrm{p}} \gamma_{kk} r_\sigma(A_k)^2 \\ &= r_\sigma(A_0)^2 + \sum_{\substack{k,l=1 \\ k \neq l}}^{\mathrm{p}} |\rho_k|\,|\rho_l|\, r_\sigma(A_k)\, r_\sigma(A_l) \\ &\quad + 2r_\sigma(A_0) \sum_{k=1}^{\mathrm{p}} |\rho_k|\, r_\sigma(A_k) \\ &\quad + \sum_{k=1}^{\mathrm{p}} (|\rho_k|^2 + \gamma_{kk})\, r_\sigma(A_k)^2 = \langle S_w s_A ; s_A \rangle. \end{aligned}$$

5. Conclusion

In this paper we have established sufficient conditions for mean square stability of discrete bilinear systems operating in a stochastic environment. Mean square stability has been defined in (D-1) in terms of the asymptotic behaviour of the state mean and correlation sequences. For the second order stochastic environment under consideration only wide sense stationarity and independence were required, thus dismissing ergodicity assumptions. The probability distributions involved were alowed to be arbitrary and unknow.

Theorems (T-1)–(T-3), which comprise the main results of the present paper, have been proved by using a preliminary stability result proposed in Lemma (L-1) for a class of deterministic nonlinear dynamical systems evolving in a Banach space. A general case has been investigated in (T-1), and two particularizations have been considered in (T-2) and (T-3). In (T-2) a model disturbance with diagonal covariance was assumed, and in (T-3) we also assumed commutativity for the system operators. The result in (T-1) generalizes the particular case proposed in [19], by extending it to a wider class of models.

The mean square stability conditions presented in (T-1)–(T-3) have been discussed in detail in remarks (R-3)–(R-5) and (R-8)–(R-12). They are based on upper bounds for norm and spectral radius of the system operators in $\mathscr{B}[\mathbb{F}'']$, weighted by model disturbance covariance coefficients. Hence they may be easily checked in practice, since they do not require analysing Liapunov operator equations.

REFERENCES

1. A. RUBERTI, A. ISIDORI, AND P. D'ALLESSANDRO, "Theory of Bilinear Dynamical Systems," CISM Course and Lectures, No. 158, Springer-Verlag, Wien, 1972.
2. R. R. MOHLER, "Bilinear Control Processes with Applications to Engineering and Medicine," Academic Press, New York, 1973.
3. C. BRUNI, G. I. PILLO AND G. KOCH, Bilinear systems: An appealing class of 'nearly linear' systems in theory and applications, *IEEE Trans. Automat. Control* **19** (1974), 334–343.
4. R. R. MOHLER AND W. J. KOLODZIEJ, An overview of bilinear systems theory and applications, *IEEE Trans. Systems Man Cybernet.* **10** (1980), 683–689.
5. R. R. MOHLER AND W. J. KOLODZIEJ, An overview of stochastic bilinear control processes, *IEEE Trans. Systems Man Cybernet.* **10** (1980), 913–919.
6. R. R. MOHLER AND A. RUBERTI (Ed.), "Theory and Applications of Variable Structure Systems," Academic Press, New York, 1972.
7. D. Q. MAYNE AND R. W. BROCKETT (Ed.), "Geometric Methods in System Theory," Reidel, Dordrecht, 1973.
8. A. RUBERTI AND R. R. MOHLER (Ed.), "Variable Systems with Applications to Economics and Biology," Lecture Notes in Economics and Mathematical Systems, Vol. 111, Springer-Verlag, Berlin, 1975.
9. R. R. MOHLER AND A. RUBERTI (Ed.), "Recent Development in Variable Structure Systems Economics and Biology," Lecture Notes in Economics and Mathematical Systems, Vol. 162, Springer-Verlag, Berlin, 1978.
10. H. KUSHNER, "Stochastic Stability and Control," Academic Press, New York, 1967.
11. J. KOZIN, A survey of stability of stochastic systems, *Automatica* **5** (1969), 95–112.

12. R. W. Brockett, Lie algebra and Lie groups in control theory, *in* "Geometric Methods in System Theory" (D. Q. Mayne and R. W. Brockett, Eds), pp. 43–82, Reidel, Dordrecht, 1973.
13. R. F. Curtain (Ed.), "Stability of Stochastic Dynamical Systems," Lecture Notes in Mathematics, Vol. 294, Springer-Verlag, Berlin, 1972.
14. J. Zabczyk, On the stability of infinite-dimensional linear stochastic systems, *in* "Probability Theory" (Z. Ciesielski, Ed.), Vol. 5, pp. 273–281, Banach Center Publication, Warsaw, 1979.
15. A. Ichikawa, Stability of semilinear stochastic evolution equations, *J. Math. Anal. Appl.* **90** (1982), 12–44.
16. J. C. Willems and G. L. Blankenship, Frequency domain stability criteria for stochastic systems, *IEEE Trans. Automat. Control* **16** (1971), 292–299.
17. J. L. Willems, Mean square stability criteria for stochastic feedback systems, *Internat. J. Systems Sci.* **4** (1973), 545–564.
18. T. Morozan, On the stability of stochastic discrete systems, *in* "Control Theory and Topics in Functional Analysis," Vol. III, pp. 225–254, Internat. Atomic Energy Agency, Vienna, 1976.
19. C. S. Kubrusly, Identification of discrete-time stochastic bilinear systems, *Internat. J. Control* **33** (1981), 291–309.
20. C. S. Kubrusly, Mean square stability for discrete bounded linear systems in Hilbert space, *SIAM J. Control Optim.* **23** (1985), 19–29.
21. W. L. Ferrar, "A Textbook of Convergence," Oxford Univ. Press, Oxford, 1938.
22. N. Dunford and J. T. Schwartz, "Linear Operators, Part I: General Theory," Wiley, New York, 1958.
23. P. R. Halmos, "A Hilbert Space Problem Book," Van Nostrand, New York, 1967.
24. S. MacLane and G. Birkhoff, "Algebra," Macmillan, New York, 1967.